

ON THE FINITE DIMENSIONALITY OF A K3 SURFACE

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Abstract. For a smooth projective surface X the finite dimensionality of the Chow motive $h(X)$, as conjectured by S.I Kimura, has several geometric consequences. For a complex surface of general type with $p_g = 0$ it is equivalent to Bloch's conjecture. The conjecture is still open for a K3 surface X which is not a Kummer surface. In this paper we prove some results on Kimura's conjecture for complex K3 surfaces. If X has a large Picard number $\rho = \rho(X)$, i.e $\rho = 19, 20$, then the motive of X is finite dimensional. If X has a non-symplectic group acting trivially on algebraic cycles then the motive of X is finite dimensional. If X has a symplectic involution i , i.e a Nikulin involution, then the finite dimensionality of $h(X)$ implies $h(X) \simeq h(Y)$, where Y is a desingularization of the quotient surface $X / \langle i \rangle$. We give several examples of K3 surfaces with a Nikulin involution such that the isomorphism $h(X) \simeq h(Y)$ holds, so giving some evidence to Kimura's conjecture in this case.

1. INTRODUCTION

For a smooth projective variety X over a field k we will denote by $A^i(X)$ the Chow group of codimension i cycles with rational coefficients and by $\mathcal{M}_{rat}(k)$ the (covariant) category of Chow motives with rational coefficients over the field k , which is is a \mathbf{Q} -linear, pseudoabelian, tensor category.

An object $M \in \mathcal{M}_{rat}(k)$ is of the form $M = (X, p, m)$, where X is a smooth projective variety over k , p a correspondence in $X \times X$ such that $p^2 = p$ and $m \in \mathbf{Z}$. We will denote by $h(X)$ the motive $(X, \Delta_X, 0)$, where Δ_X is the diagonal in $X \times X$. If X and Y are smooth (irreducible) projective varieties over k then

$$Hom_{\mathcal{M}_{rat}(k)}(h(X), h(Y)) = A^{dim X}(X \times Y)$$

where $A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbf{Q}$.

We will consider a classical Weil cohomology theory H^* with coefficients in a field K of characteristic 0 which induces a tensor functor $H^* : \mathcal{M}_{rat} \rightarrow Vect_K^{gr}$ such that $H^i((X, p, m) = p^* H^{i-2m}(X, K)$ (see [KMP 1.4]). If $char\ k = 0$ homological equivalence does not depend on the choice of H^* . By replacing rational equivalence with homological equivalence we get the category $\mathcal{M}_{hom}(k)$ of homological motives.

For an object $M \in \mathcal{M}_{rat}(k)$, one defines the exterior power $\wedge^n M \in \mathcal{M}_{rat}(k)$ (and similarly in \mathcal{M}_{hom}) and the symmetric power $S^n M$. (see [Ki]). A motive M is *finite dimensional* if it can be decomposed as $M = M^+ \oplus M^-$ with M^+ evenly finite dimensional , i.e such that $\wedge^n M = 0$ for some $n > 0$ and M^- oddly finite dimensional, i.e such that $S^n M = 0$ for $n > 0$.

S.I.Kimura and O'Sullivan (se [Ki]) have conjectured that all the motives in $\mathcal{M}_{rat}(k)$ are finite dimensional. The conjecture is known for curves, for abelian varieties and for some surfaces: rational surfaces, Godeaux surfaces, Kummer surfaces, surfaces with $p_g = 0$ which are not of general type, surfaces isomorphic to a quotient $(C \times D)/G$, where C and D are curves and G is a finite group. It is also known for Fano 3-folds (see[G-G]). In all these known cases the motive $h(X)$ lies in the tensor subcategory of $\mathcal{M}_{rat}(k)$ generated by abelian varieties.

If $M = h(X)$ is the motive of a surface then the finite dimensionality of M is equivalent to the vanishing of $\wedge^n t_2(X)$ for some $n > 0$, where $t_2(X)$ is the *transcendental part* of $h(X)$. This follows from the existence of a refined Chow-Künneth decomposition for the motive $h(X)$ of a surface

$$h(X) = \mathbf{1} \oplus h_1(X) \oplus h_{alg}^2(X) \oplus t_2(X) \oplus h_3(X) \oplus \mathbf{L}^2$$

where $\mathbf{1}$ is the motive of a point and \mathbf{L} is the Lefschetz motive. (see [KMP]). In the above decomposition all the summands, but possibly $t_2(X)$, are finite dimensional because they lie in the subcategory of $\mathcal{M}_{rat}(k)$ generated by abelian varieties. Therefore the information necessary to study the above conjecture for a surface X is concentrated in the transcendental part of the motive $t_2(X)$. More precisely, according to Murre's Conjecture (see [Mu]), or equivalently to Bloch-Beilinson's conjecture (see [J]) and to Kimura's Conjecture the following results should hold for a surface X

- (a) The motive $t_2(X)$ is evenly finite dimensional;
- (b) $h(X)$ satisfies the Nilpotency conjecture , i.e every homologically trivial endomorphism of $h(X)$ is nilpotent ;
- (c) Every homologically trivial correspondence in $CH^2(X \times X)_{\mathbf{Q}}$ acts trivially on the Albanese kernel $T(X)$;

d) The endomorphism group of $t_2(X)$ (tensored with \mathbf{Q}) has finite rank (over a field of characteristic 0).

By a result of S.Kimura in [Ki] , (a) implies (b).

If X is a complex surface of general type with $p_g(X) = 0$, Bloch's conjecture asserts that $A_0(X) \simeq \mathbf{Q}$. Then

$$(a) \iff A_0(X) = \mathbf{Q} \iff t_2(X) = 0$$

(see [G-P]).

A case where all the above conjectures are still unknown is that of a complex K3 surface which is not a Kummer surface.

The aim of this paper is to prove some results about the finite dimensionality of $h(X)$ in the case X is a K3 surface over \mathbf{C} .

Note that a result by Y.Andre' in [A 10.2.4.1] implies that the motive of a K3 surface is isomorphic to the motive of an abelian variety in a suitable category of *motivated motives*. Under the standard conjecture $B(X)$ this category coincides with \mathcal{M}_{hom} (see [A p.100]. Therefore Andre's result suggests that the Chow motive of every K3 surface can be expressed in terms of the motives of abelian varieties.

In §2 we consider the case of a projective surface X with an involution σ and the desingularization Y of the quotient surface $X / \langle \sigma \rangle$. Corollary 1 gives necessary and sufficient conditions on σ for the existence of an isomorphism $t_2(X) \simeq t_2(Y)$ and for $t_2(Y) = 0$. In particular this result applies to a complex surface of general type X with $p_g(X) = 0$ and an involution σ for which $t_2(Y) = 0$.

In §3 we apply the results in §2 to the case of a complex K3 surface X with an involution σ . If σ is symplectic , i.e σ is a *Nikulin involution*, then the finite dimensionality of $h(X)$ implies the isomorphism $h(X) \simeq h(Y)$, see Theorem 3. If the rank of the Neron- Severi group of X is 19 or 20, then $h(X)$ is finite dimensional (Theorem 2) . If σ is not symplectic then $t_2(Y) = 0$, with $Y = X / \langle \sigma \rangle$, hence $t_2(X) \neq t_2(Y)$, see Remark 3. If a K3 surface X has a non-symplectic group acting trivially on algebraic cycles then the motive of X is finite dimensional (Corollary 2). Note that, in all the cases where we can show that the motive $h(X)$ of a K3 surface is finite dimensional, $h(X)$ lies in the tensor subcategory of $\mathcal{M}_{rat}(k)$ generated by abelian varieties.

In §4 , using the results in [VG-S], we describe several examples of K3 surfaces , with a Nikulin involution i and Picard rank 9, such that $t_2(X) \simeq t_2(Y)$. We also show (see Theorem 7) that the same result holds if the K3 surface X has an elliptic fibration $X \rightarrow \mathbf{P}^1$ with a section. This gives some evidence to Kimura's conjecture for a K3 surface with a symplectic involution.

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2. SURFACES WITH AN INVOLUTION

In this section we prove some results on the transcendental part $t_2(X)$ of the motive of a surface X , with an involution σ .

We first note that, if X is a smooth projective variety over a field k , and G is a finite group acting on X , then the theory of correspondences can be extended to $Y = X/G$, if one uses rational coefficients in the Chow groups (see [Fu 16.1.13]). In particular this holds if $G = \langle \sigma \rangle$, where σ is an involution.

Let X be a smooth irreducible projective surface (over any field k) with a refined Chow-Künneth decomposition $\sum_{0 \leq i \leq 4} h_i(X)$ where $h_2(X) = h_2^{alg}(X) + t_2(X)$ and $t_2(X) = (X, \pi_2^{tr}, 0)$, see [KMP 2.2]. Here

$$\pi_2^{alg}(X) = \sum_{1 \leq h \leq \rho} \frac{[D_h \times D_h]}{D_h^2}$$

where $\{D_h\}$ is an orthogonal basis of $NS(X) \otimes \mathbf{Q}$ and $\rho = \text{rank } NS(X)$. The map

$$(1) \quad \Psi_X : A^2(X \times X) \rightarrow \text{End}_{\mathcal{M}_{rat}}(t_2(X))$$

defined by $\Psi_X(\Gamma) = \pi_2^{tr} \circ \Gamma \circ \pi_2^{tr}$ yields an isomorphism (see [KMP 4.3])

$$A^2(X \times X) / \mathcal{J}(X) \simeq \text{End}_{\mathcal{M}_{rat}}(t_2(X))$$

where $\mathcal{J}(X)$ is the ideal of $A^2(X \times X)$ generated by the classes of correspondences which are not dominant over X by either the first or the second projection. Let $k(X)$ be the field of rational functions and let $T(X_{k(X)})$ be the Albanese kernel of $X_{k(X)}$, i.e the kernel of the Abel-Jacobi map $A_0(X_{k(X)}) \rightarrow \text{Alb}_X(k(X) \otimes \mathbf{Q})$. Let $\tau_X : A^2(X \times X) \rightarrow T(X_{k(X)})$ be the map

$$\tau_X(Z) = (\pi_2^{tr} \circ Z \circ \pi_2^{tr})(\xi)$$

with ξ the generic point of X . Then τ_X induces an isomorphism (see [KMP 5.10])

$$\text{End}_{\mathcal{M}_{rat}}(t_2(X)) \simeq \frac{T(X_{k(X)})}{H_{\leq 1} \cap T(X_{k(X)})}$$

Here $H_{\leq 1}$ is the subgroup of $A_0(X_{k(X)})$ generated by the subgroups $A_0(X_L)$, where L runs over the subfields of $k(X)$ containing k and which are of transcendence degree ≤ 1 over k . If $q(X) = 0$ then X has no odd

cohomology, $Alb_X(k) = 0$ and in the Chow -Künneth decomposition we have $h_1(X) = h_3(X) = 0$. Therefore $A_0(X_{k(X)})_0 = T(X_{k(X)})$ and $T(X) = A_0(X)_0$, where $A_0(X)_0$ is the group of 0-cycles of degree 0. By [KMP 5.10] we have

$$H_{\leq 1} \cap T(X_{k(X)}) = T(X)$$

Hence , for a surface X with $q(X) = 0$, the map τ_X yields an isomorphism

$$(2) \quad End_{\mathcal{M}_{rat}}(t_2(X)) \simeq \frac{A_0(X_{k(X)})}{A_0(X)}$$

where the class $[\xi]$ in $\frac{A_0(X_{k(X)})}{A_0(X)}$ of the generic point $[\xi]$ of X corresponds to the identity of the ring $End_{\mathcal{M}_{rat}}(t_2(X))$

The definition of the map Ψ_X in (1) can be extended to the case of two smooth projective surfaces X and X' as in [KMP 7.4]

$$\Psi_{X,X'} : A^2(X \times X') \rightarrow Hom_{\mathcal{M}_{rat}}(t_2(X), t_2(X'))$$

and the following functorial relation holds

$$(3) \quad \Psi_{X,X''}(\Gamma' \circ \Gamma) = \Psi_{X',X''}(\Gamma') \circ \Psi_{X,X'}(\Gamma)$$

where X, X', X'' are smooth projective surfaces, $\Gamma \in A^2(X \times X')$ and $\Gamma' \in A^2(X' \times X'')$. The proof of (3) immediately follows by taking refined Chow-Künneth decompositions of the motives $h(X)$, $h(X')$, $h(X'')$ and writing the elements in $Hom_{\mathcal{M}_{rat}}(h(X), h(X'))$, and $Hom_{\mathcal{M}_{rat}}(h(X'), h(X''))$ as lower triangular matrices defined by these decompositions, as in [KMP p.163]. Applying Ψ corresponds to taking appropriate diagonal entries of such lower triangular matrices.

Lemma 1. *Let X and Y are smooth projective surfaces and let $f : X \rightarrow Y$ be a finite morphism. Then f induces homomorphisms $\bar{f}_* : End_{\mathcal{M}_{rat}}(t_2(X)) \rightarrow End_{\mathcal{M}_{rat}}(t_2(Y))$ and $\bar{f}^* : End_{\mathcal{M}_{rat}}(t_2(Y)) \rightarrow End_{\mathcal{M}_{rat}}(t_2(X))$.*

Proof. The maps $\Psi_X : A^2(X \times X) \rightarrow End_{\mathcal{M}_{rat}}(t_2(X))$ and $\Psi_Y : A^2(Y \times Y) \rightarrow End_{\mathcal{M}_{rat}}(t_2(Y))$ give rise to the following commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow \mathcal{J}(X) & \longrightarrow & A^2(X \times X) & \xrightarrow{\Psi_X} & End_{\mathcal{M}_{rat}}(t_2(X)) & \longrightarrow & 0 \\ & & \downarrow (f \times f)_* & & \downarrow \bar{f}_* & & \\ 0 \rightarrow \mathcal{J}(Y) & \longrightarrow & A^2(Y \times Y) & \xrightarrow{\Psi_Y} & End_{\mathcal{M}_{rat}}(t_2(Y)) & \longrightarrow & 0 \\ & & \downarrow (f \times f)^* & & \downarrow \bar{f}^* & & \\ 0 \rightarrow \mathcal{J}(X) & \longrightarrow & A^2(X \times X) & \xrightarrow{\Psi_X} & End_{\mathcal{M}_{rat}}(t_2(X)) & \longrightarrow & 0 \end{array}$$

where the map $(f \times f)_*$ sends a correspondence $Z \in A^2(X \times X)$ to $\Gamma_f \circ Z \circ \Gamma_f^t$ and the map $f \times f^*$ sends a correspondence $Z' \in A^2(Y \times Y)$ to $\Gamma_f^t \circ Z' \circ \Gamma_f$. It is easy to see that these maps send the ideal $\mathcal{J}(X)$ to $\mathcal{J}(Y)$ and $\mathcal{J}(Y)$ to $\mathcal{J}(X)$ respectively, thus yielding the diagram above. \square

Proposition 1. Let X be a smooth projective surface with an involution σ , such that the quotient surface $Y = X / \langle \sigma \rangle$ is smooth. Let ξ denote the generic point of X , η the generic point of Y and let $[\xi] = \Psi_X(\Delta_X) \in \text{End}_{\mathcal{M}_{rat}}(t_2(X))$, $[\eta] = \Psi_Y(\Delta_Y) \in \text{End}_{\mathcal{M}_{rat}}(t_2(Y))$. Set $\alpha = \Psi_X(1 \times \sigma)\Delta_X = \Psi_X(\Gamma_\sigma) = \bar{\sigma}([\xi])$. Then the map $f : X \rightarrow Y$ satisfies:

- (i) $1/2(\Gamma_f \circ \Gamma_f^t) = \Delta_Y$, $\bar{f}_*([\xi]) = \bar{f}_*(\alpha) = 2[\eta]$ and $(\alpha)^2 = [\xi]$.
- (ii) $\bar{f}^*([\eta]) = [\xi] + \alpha$ and $\bar{f}^*(\bar{f}_*([\xi])) = 2[\xi] + 2\alpha$.
- (iii) Let $p = 1/2(\Gamma_f^t \circ \Gamma_f)$; then $p \circ p = p$, $\Psi_X(p) = 1/2([\xi] + \alpha)$ and $\Psi_X(\Delta_X - p) = 1/2([\xi] - \alpha)$. Hence $[\xi] = 1/2([\xi] + \alpha) + 1/2([\xi] - \alpha)$.

Proof. Regard the diagonals Δ_X and Δ_Y as cycles in $A^2(X \times X)$ and $A^2(Y \times Y)$. Then $\bar{f}_*([\xi])$ is the image under Ψ_Y of $\Gamma_f \circ \Gamma_f^t = 2\Delta_Y$. Thus $\bar{f}_*([\xi]) = 2\Psi_Y(\Delta_Y) = 2[\eta]$ and we also have

$$p \circ p = (1/4)\Gamma_f^t \circ (2\Delta_Y) \circ \Gamma_f = 1/2(\Gamma_f^t \circ \Gamma_f) = p$$

Since α is the image of $(1 \times \sigma)\Delta_X$, and $(1 \times \sigma)\Delta_X \cdot (1 \times \sigma)\Delta_X = \Delta_X$ we have $\alpha^2 = [\xi]$. Since $\Gamma_f \cdot (1 \times \sigma)\Delta_X = \Gamma_f$, the correspondence $(1 \times \sigma)\Delta_X$ also maps to Δ_Y , so $f_*(\alpha) = 2[\eta]$. This establish (i) and (ii) follows immediately. Part (iii) follows from (ii) and $p \circ p = p$. \square

Let X be a smooth projective surface and let σ be an involution on X . Let k be the number of isolated fixed points of σ and let D the 1-dimensional part of the fixed-point locus. The divisor D is smooth (possibly empty). Let \tilde{X} be the blow-up of of the set of isolated fixed points. Then the involution σ lifts to an involution on \tilde{X} (which we will still denote by σ). The quotient $Y = \tilde{X} / \langle \sigma \rangle$ is a desingularization of $X / \langle \sigma \rangle$. Y has k disjoint nodal curves C_1, \dots, C_k . The map $X \rightarrow X / \langle \sigma \rangle$ induces a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\beta} & X \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & X / \langle \sigma \rangle \end{array}$$

Since $t_2(-)$ is a birational invariant for smooth projective surfaces the maps $\beta : \tilde{X} \rightarrow X$ and $f : \tilde{X} \rightarrow Y$ induce a morphism

$$\theta : t_2(\tilde{X}) = t_2(X) \rightarrow t_2(Y)$$

Corollary 1. *Let X, \tilde{X}, Y be as in the diagram above. Then*

- (i) $\theta : t_2(X) \rightarrow t_2(Y)$ is the projection onto a direct summand.
- (ii) θ is an isomorphism iff $\Psi_X(\Gamma_\sigma) = id_{t_2(X)}$, i.e. iff $\bar{\sigma}([\xi]) = [\xi]$ in $End_{\mathcal{M}_{rat}}(t_2(X))$.
- (iii) If $q(X) = 0$ the conditions of (ii) are equivalent to $A_0(X)_0^\sigma = A_0(X)_0$.
- (iv) $t_2(Y) = 0 \iff \Psi_X(\Gamma_\sigma) = -id_{t_2(X)} \iff \bar{\sigma}([\xi]) = -[\xi]$ in $End_{\mathcal{M}_{rat}}(t_2(X))$. If $q(X) = 0$ this is equivalent to $A_0(X)_0^\sigma = 0$.

Proof. Since $\Psi_{\tilde{X},X}(\Gamma_\beta)$ is an isomorphism and $\theta = \Psi_{\tilde{X},Y}(\Gamma_f) \circ \Psi_{\tilde{X},X}(\Gamma_\beta)^{-1}$ it is enough, after replacing X by \tilde{X} , to prove the Corollary under the assumption $\tilde{X} = X$. Then $\theta = \Psi_{X,Y}(\Gamma_f)$. From Proposition 1 we get that Γ_f has a right inverse $1/2(\Gamma_f^t)$ and $2p = \Gamma_f^t \circ \Gamma_f = \Delta_X + (1 \times \sigma)\Delta_X = \Delta_X + \Gamma_\sigma$. It follows from the functoriality of Ψ in (3) that, if $t_2(X)^+$ and $t_2(X)^-$ are the direct summands of $t_2(X)$ on which the involution $\Psi_X(\Gamma_\sigma)$ acts respectively as $+1$ or -1 , then the restriction of θ to $t_2(X)^-$ is 0 and to $t_2(X)^+$ is an isomorphism. This gives (i). Also θ is an isomorphism iff $t_2(X)^- = 0$ which is equivalent to $\Psi_X(\Gamma_\sigma)$ being the identity in $End_{\mathcal{M}_{rat}}(t_2(X))$. This gives (ii). If $q(X) = 0$ then $A_0(X)_0 = T(X)$ and we have a canonical isomorphism

$$Hom_{\mathcal{M}_{rat}}(\mathbf{1}, t_2(X)) \simeq A_0(X)_0$$

which is compatible with the action of correspondences. Hence, by taking the action of $\Psi_X(\Gamma_\sigma)$ on $t_2(X)$ we get

$$Hom_{\mathcal{M}_{rat}}(\mathbf{1}, t_2(X)^-) \simeq A_0(X)_0^-$$

Therefore $\Psi_X(\Gamma_\sigma)$ acts as the identity on $t_2(X)$ iff $A_0(X)_0^- = A_0(t_2(X)^-) = 0$. Since $A_i(t_2(X)) = 0$ for $i \neq 0$, we have $A_i(M) = 0$ for all i , where $M = t_2(X)^-$. It follows that $M = 0$ (see [C-G Lemma 1]. This proves (iii).

Clearly $t_2(Y) = 0$ is equivalent to $\bar{\sigma}([\xi]) = -[\xi] \in End_{\mathcal{M}_{rat}}(t_2(X))$. Since the cycle class $[\xi]$ corresponds to the identity of $End_{\mathcal{M}_{rat}}(t_2(X))$ under the isomorphism in (2), $\Psi_X(\Gamma_\sigma)$ acts as -1 on $t_2(X)$. Let $q(X) = 0$: then, by the same argument as in the proof of (iii) we get $A_0(X)_0^\sigma = 0$. This gives (iv). \square

F. Severi in [Sev] has introduced the notions of *valence* and *indices* of a correspondence $T \in A^n(X \times X)$, where X is a smooth projective

variety of dimension n . In the case when X is a surface, Severi related these notions to the computation of the degree of the cycle $T \cdot \Delta_X$.

Definition 1. Let X be a smooth projective variety of dimension n . A correspondence $T \in A^n(X \times X)$ has valence 0 if it belongs to the ideal of degenerate correspondences, i.e the ideal generated by correspondences of the form $[V \times W]$, with V, W proper subvarieties of X . A correspondence Γ has valence v if $T = \Gamma + v\Delta_X$ has valence 0. If $T = T_1 + T_2$ in $A^d(X \times X)$ and T_1, T_2 have valences v_1, v_2 then T has valence $v_1 + v_2$. If the correspondences T and T' have valences v and v' then $v(T \circ T') = -v(T) \cdot v(T')$; see[Fu 16 1.5.]. It follows that if p is a projector in $A^2(X \times X)$ which has a valence, then $v(p)$ is either 0 or -1.

The indices of a correspondence T are the numbers $\alpha(T) = \deg(T \cdot [P \times X])$ and $\beta(T) = \deg(T \cdot [X \times P])$, where P is any rational point on X ; see [Fu, 16 1.4]. The indices are additive in T and $\beta(T) = \alpha(T^t)$.

Theorem 1. *Let X be a smooth projective surface with an involution σ and let Y be the desingularization of $X / \langle \sigma \rangle$. Assume $p_g(X) > 0$ and let $\Gamma_\sigma = (1 \times \sigma)\Delta_X$. If the correspondence Γ_σ has a valence, then*

$$t_2(Y) = 0 \iff v(\Gamma_\sigma) = 1 ; \quad \theta : t_2(X) \xrightarrow{\sim} t_2(Y) \iff v(\Gamma_\sigma) = -1$$

Proof. : Let $[\xi] = 1/2([\xi] + \alpha) + 1/2([\xi] - \alpha)$ be the splitting in $\text{End}_{\mathcal{M}_{rat}}(t_2(X))$ coming from Proposition 1, with $\alpha = \bar{\sigma}([\xi]) = \Psi_X(\Gamma_\sigma)$. If the correspondence Γ_σ has a valence then also the projector $p = 1/2(\Delta_X + (1 \times \sigma)\Delta_X) = 1/2(\Delta_X + \Gamma_\sigma)$ has a valence and $v(p)$ is either 0 or -1. Since $v(\Delta_X) = -1$ we have

$$v(p) = 0 \iff v(\Gamma_\sigma) = 1 ; \quad v(p) = -1 \iff v(\Gamma_\sigma) = -1$$

Suppose $v(\Gamma) = 1$: then $v(p) = 0$ i.e p belongs to the ideal of degenerate correspondences, which is contained in $\text{Ker } \Psi_X$. From $\Psi_X(p) = 0$ we get $1/2([\xi] + \Psi_X(\Gamma_\sigma)) = 0$ hence $\Psi_X(\Gamma_\sigma) = -id_{t_2(X)}$. From Corollary (iv) we get $t_2(Y) = 0$. Conversely if $t_2(Y) = 0$, then $\Psi_X(\Gamma_\sigma) = -id_{t_2(X)}$ hence $\Psi_X(p) = 0$ and we get $v(p) = 0$.

If $\theta : t_2(X) \rightarrow t_2(Y)$ is an isomorphism then, by Corollary 1 (ii) $\Psi_X(\Gamma_\sigma) = id_{t_2(X)}$. By the same argument as before we get $v(\Gamma_\sigma) = -1$. \square

Remark 1. The assumption $p_g(X) > 0$ in Theorem 1 is necessary in order to have a uniquely defined valence for Γ_σ . If $p_g(X) = 0$ and X satisfies Bloch's conjecture then, by the results in [B-S], $v(\Delta_X) = 0$, hence the correspondence Δ_X has 2 different valences; namely 0 and -1. Note that, for a surface X , a correspondence Γ can have 2 different valences v and v' only if $p_g(X) = 0$. This was first observed by Severi

in [Sev p.761]). In fact then the multiple $(v - v')$ of the diagonal Δ_X belongs to the ideal of degenerate correspondences and this implies that $\Psi_X(\Delta_X) = 0$ in $End_{\mathcal{M}_{rat}}(t_2(X))$. Therefore the identity map is 0 in $End_{\mathcal{M}_{rat}}(t_2(X))$. Hence $t_2(X) = 0$ and this may occur only if $p_g(X) = 0$.

3. COMPLEX K3 SURFACES

A smooth (irreducible) projective K3 surface X over \mathbf{C} is a regular surface (i.e $q(X) = 0$), therefore it has a refined Chow-Künneth decomposition (see [KMP 2.2]) of the form $h(X) = \sum_{0 \leq i \leq 4} h_i(X)$ with $h_1(X) = h_3(X) = 0$. Moreover $h_2(X) = h_2^{alg}(X) + t_2(X)$, where $t_2(X) = (X, \pi_2^{tr}, 0)$ and $h_2^{alg}(X) \simeq \mathbf{L}^{\oplus \rho(X)}$. Here $\rho(X)$ is the rank of the $NS(X)_{\mathbf{Q}} = (PicX)_{\mathbf{Q}}$ so that $1 \leq \rho \leq 20$. Moreover

$$H^i(t_2(X)) = 0 \text{ for } i \neq 2 ; H^2(t_2(X)) = \pi_2^{tr} H^2(X, \mathbf{Q}) = H_{tr}^2(X, \mathbf{Q}),$$

$$A_i(t_2(X)) = \pi_2^{tr} A_i(X) = 0 \text{ for } i \neq 2 ; A_0(t_2(X)) = T(X),$$

where $T(X)$ is the Albanese Kernel. Since $q(X) = 0$, we also have $T(X) = A_0(X)_0$ (0-cycles of degree 0) and

$$\dim H^2(X) = b_2(X) = 22 ; \dim H_{tr}^2(X) = b_2(X) - \rho$$

A Nikulin involution i of a complex K3 surface X is a symplectic automorphism of order 2 , i.e such that $i^* \omega = \omega$ for all $\omega \in H^{2,0}(X)$. A K3 surface X with a Nikulin involution has rank $\rho(X) \geq 9$. The Neron-Severi group $NS(X)$ contains a primitive sublattice isomorphic to $E_8(-2)$ where E_8 is the unique even unimodular positive defined lattice of rank 8 (see [Mor p.106]). Here, if L is a lattice and m is an integer, $L(m)$ denotes same free \mathbf{Z} -module L with a form which has been altered by multiplication by m , that is $b_{L(m)}(x, y) = m(b_L(x, y))$, where $b_L(x, y)$ is the \mathbf{Z} -valued symmetric bilinear form of L . By T_X we will denote the transcendental lattice of X , i.e $T_X = NS(X)^{\perp} \subset H^2(X, \mathbf{Z})$. For any K3 surface with a Nikulin involution i there is an isomorphism

$$H^2(X, \mathbf{Z}) \simeq U^3 \oplus E_8(-1) \oplus E_8(-1)$$

where U is the hyperbolic plane, such that i^* acts as follows

$$i^*(u, x, y) = (u, y, x)$$

The invariant sublattice is $H^2(X, \mathbf{Z})^i \simeq U^3 \oplus E_8(-2)$ and $(H^2(X, \mathbf{Z})^i)^{\perp} \simeq E_8(-2)$. Since $i^* \omega = \omega$ for all $\omega \in H^{2,0}(X)$ we also have $(H^2(X, \mathbf{Z})^i)^{\perp} \subset NS(X)$ (see [VG-S 2.1]). Therefore the involution i acts as the identity on $H_{tr}^2(X, \mathbf{Q})$.

Let $X \rightarrow X / \langle i \rangle$ be the quotient map. The surface $X / \langle i \rangle$ has 8 ordinary double points Q_1, \dots, Q_8 corresponding to the 8 fixed

points P_1, \dots, P_8 of the involution i on X . The minimal model Y of $X / \langle i \rangle$ is a K3 surface, hence $p_g(Y) > 0$. In the following we will always consider the *standard diagram* for a K3 surface with a Nikulin involution i (see [Mor sec. 3])

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\beta} & X \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & X / \langle i \rangle \end{array}$$

\tilde{X} is the blow up of X at the points P_1, \dots, P_8 with exceptional divisors $\beta^{-1}(P_j) = E_j$. The Nikulin involution extends to an involution i on \tilde{X} and $Y = \tilde{X} / \langle i \rangle$. f is a double cover branched on the divisor $\sum_{1 \leq j \leq 8} C_j$ where $C_j = f(E_j)$ are disjoint smooth irreducible rational curves corresponding to the points Q_1, \dots, Q_8 . Therefore $1/2(\sum_j C_j) \in NS(Y)$. The map $f_* \circ \beta^*$ induces an isomorphism of rational Hodge structures

$$T_{\tilde{X}} \otimes \mathbf{Q} \simeq T_X \otimes \mathbf{Q} \simeq T_Y \otimes \mathbf{Q}$$

where T_X and T_Y are the transcendental lattices. In particular the vector spaces $H_{tr}^2(X, \mathbf{Q})$ and $H_{tr}^2(Y, \mathbf{Q})$ have the same dimension, so that $22 - \rho(X) = 22 - \rho(Y)$.

Suppose conversely that a K3 surface Y admits an even set of k disjoint rational curves C_1, \dots, C_k : this means that there exists a $\delta \in Pic Y$ such that

$$C_1 + \dots + C_k \sim 2\delta$$

This is equivalent to the existence of a double cover X of Y branched on $C_1 + \dots + C_k$. Then, by [N 1], $k = 0, 8, 16$. If $k = 16$ then X is birational to an abelian surface A and Y is the Kummer surface of A . Therefore the motives $h(X)$ and $h(Y)$ are finite dimensional and $t_2(A) \simeq t_2(X) \simeq t_2(Y)$ (see [KMP 6.13]). If $k = 8$ then X is a K3 surface, Y is the desingularization of the quotient of X by a Nikulin involution i . Hence Y is a K3 surface.

Theorem 2. *Let X be a smooth projective K3 surface over \mathbf{C} with $\rho(X) = 19, 20$. Then the motive $h(X) \in \mathcal{M}_{rat}(\mathbf{C})$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties.*

Proof. By [Mor 6.4] X admits a Shioda-Inose structure, i.e there is a Nikulin involution i on X such that the desingularization Y of the quotient surface $X / \langle i \rangle$ is a Kummer surface, associated to an abelian surface A . Hence $h(Y)$ is finite dimensional. The rational map

$f : X \rightarrow Y$ induces a splitting $t_2(X) \simeq t_2(Y) \oplus N$. Since $t_2(Y)$ is finite dimensional we are left to show that $N = 0$. From Corollary 1 the vanishing of N is equivalent to $A_0(X)_0^i = A_0(X)_0$. By [Mor 6.3 (iv)] the Neron-Severi group of X contains the sublattice $E_8(-1)^2$. Hence by the results in [Huy 6.3, 6.4], the symplectic automorphism i acts as the identity on $A_0(X)$. From Corollary 1 we get $t_2(X) = t_2(Y)$. By [KMP 6.13] $t_2(Y) = t_2(A)$; therefore $h(X)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties. \square

Remark 2. Note that by [Mo 2.10 (i), 4.4(i)] there exist K3 surfaces with $\rho(X) = 19, 20$ which are not Kummer surfaces.

Next we show that for every K3 surface with a Nikulin involution the finite dimensionality of $h(X)$ implies $h(X) \simeq h(Y)$.

Lemma 2. *Let X be a K3 surface over \mathbf{C} with a Nikulin involution i and let Y be a desingularization of the quotient surface $X/\langle i \rangle$. Let $e(-)$ be the topological Euler characteristic. Then we have*

$$e(X) + t + 2 + 2k = 2e(Y)$$

where t is the trace of the involution i on $H^2(X, \mathbf{C})$ and $k = 8$ is the number of the isolated fixed points of i . Therefore $\rho(X) = \rho(Y)$ and $t = 6$

Proof. : We use the same argument as in [D-ML-P 4.2]. Since i has only isolated fixed points from the topological fixed point formula we get

$$e(X) + t + 2 = 2e(Y) - 2k$$

Since X and Y are both K3 surfaces we have $e(X) = e(Y) = 24$. Therefore, we get $t = 6$. Since $\dim H_2^{tr}(X) = \dim H_2^{tr}(Y)$ and $b_2(X) = b_2(Y) = 22$, we have $\rho(X) = \rho(Y)$ \square

Theorem 3. *Let X be K3 surface with a Nikulin involution i . If $h(X)$ is finite dimensional then $h(X) \simeq h(Y)$.*

Proof. Y is a K3 surface and we have $t_2(\tilde{X}) = t_2(X)$ because $t_2(-)$ is a birational invariant for surfaces. Also

$$H_{tr}^2(X) \simeq H_{tr}^2(\tilde{X}) \simeq H_{tr}^2(Y)$$

because the Nikulin involution acts trivially on $H_{tr}^2(X)$. Let t be the trace of the involution σ on the vector space $H^2(X, \mathbf{C})$. From Lemma 2 we get $t = 6$. The involution i acts trivially on $H_{tr}^2(X)$ and $H_{tr}^2(X)$ is a subvector space of $H^2(X, \mathbf{C})$ of dimension $22 - \rho$. Therefore the trace of the action of i on $NS(X) \otimes \mathbf{C}$ equals $\rho - 16$. Since the only eigenvalues

of an involution are +1 and -1 we can find an orthogonal basis for $NS(X) \otimes \mathbf{C}$ of the form $H_1, \dots, H_r; D_1, \dots, D_8$, with $r = \rho - 8 \geq 1$ such that $i_*(H_j) = H_j$ and $i_*(D_l) = -D_l$. Then $NS(\tilde{X}) \otimes \mathbf{C}$ has a basis of the form $E_1, \dots, E_8; H_1, \dots, H_r; D_1, \dots, D_8$, where E_h , for $1 \leq h \leq 8$ are the exceptional divisors of the blow up $\tilde{X} \rightarrow X$. The set of $r + 8 = \rho$ divisors $f_*(E_h) = C_k$, for $1 \leq h \leq 8$ and $f_*(H_j) \simeq H_j$, for $1 \leq j \leq r$ gives an orthogonal basis for $NS(Y) \otimes \mathbf{Q}$. Since $q(X) = q(Y) = q(\tilde{X}) = 0$ we can find Chow-Künneth decompositions for $h(X)$, $h(\tilde{X})$ and $h(\tilde{Y})$ of the form

$$h(X) = \mathbf{1} \oplus h_2^{alg}(X) \oplus t_2(X) \oplus \mathbf{L}^2 \simeq \mathbf{1} \oplus \mathbf{L}^{\otimes \rho} \oplus t_2(X) \oplus \mathbf{L}^2$$

$$h(\tilde{X}) = \mathbf{1} \oplus h_2^{alg}(\tilde{X}) \oplus t_2(X) \oplus \mathbf{L}^2 \simeq h(X) \oplus \mathbf{L}^{\oplus 8}$$

$$h(Y) = \mathbf{1} \oplus h_2^{alg}(Y) \oplus t_2(Y) \oplus \mathbf{L}^2 \simeq \mathbf{1} \oplus \mathbf{L}^{\otimes \rho} \oplus t_2(Y) \oplus \mathbf{L}^2$$

where $\rho(X) = \rho(Y) = \rho$. By Corollary 1 we have $t_2(X) \simeq t_2(Y) \oplus N$ for some $N \in \mathcal{M}_{rat}$. Since $h(X)$ is finite dimensional also N is finite dimensional. Since $H(X)$ and $H(Y)$ are isomorphic as graded vector spaces $H(N) = 0$. By [Ki 7.3] $N = 0$. Therefore $h(X) \simeq h(Y)$. \square

Theorem 4. *Let X be a K3 surface with a Nikulin involution i . Then the following conditions are equivalent:*

- (i) *the correspondence $\Gamma_i = (1 \times i)\Delta_X$ has a valence.*
- (ii) *$\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$.*
- (iii) *$\bar{i}([\xi]) = [\xi]$ in $End_{\mathcal{M}_{rat}}(t_2(X))$.*
- (iv) *i acts as the identity on $A_0(X)_0$*

Proof. Let

$$\Delta_X = 1/2(\Delta_X + (1 \times i)\Delta_X) + 1/2(\Delta_X - (1 \times i)\Delta_X)$$

as in Proposition 1 and let $\Gamma_i = (1 \times i)\Delta_X$. If Γ_i has a valence then also the projector $q = 1/2(\Delta_X - (1 \times i)\Delta_X)$ has a valence and $v(q)$ is either 0 or -1. Suppose that $v(1/2(\Delta_X - (1 \times i)\Delta_X)) = -1$; then the correspondence $(1 \times i)\Delta_X$ has valence 1. From Theorem 1 we get $t_2(Y) = 0$ hence a contradiction because Y is a K3 surface. Therefore $v(1/2(\Delta_X - (1 \times i)\Delta_X)) = 0$, so that $v(1 \times i)\Delta_X = v(\Delta_X) = -1$. By Theorem 1 $\theta : t_2(X) \rightarrow t_2(Y)$ is an isomorphism. Therefore (i) \Rightarrow (ii). Conversely if $\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$ then by Corollary 1 (ii) $\Psi_X(\Delta_X - \Gamma_i) = 0$, hence $\Delta_X - \Gamma_i \in Ker \Psi_X$. Since $q(X) = 0$ $Ker \Psi_X$ coincides with the ideal of degenerate correspondences. This proves (i).

The equivalences (ii) \Leftrightarrow (iii) and (iii) \Leftrightarrow (iv) come from Corollary 1. \square

Remark 3. Let σ is an involution on a K3 surface X which is not symplectic, i.e $\sigma^*(\omega) = -\omega$, where ω is a generator of the vector space $H^{2,0}(X)$. By [Zh 1.2] if $X^\sigma = \emptyset$ the quotient surface $Y = X / \langle \sigma \rangle$ is an Enriques surface, while Y is a rational surface if $X^\sigma \neq \emptyset$. In any case the motive $h(Y)$ has no transcendental part. Therefore $t_2(Y) = 0$ and $t_2(X) \neq t_2(Y)$, because $t_2(X) \neq 0$ for a K3 surface. From the identity in $End_{\mathcal{M}_{rat}}(t_2(X))$

$$[\xi] = 1/2([\xi] + \bar{\sigma}([\xi]) + 1/2([\xi] - \bar{\sigma}([\xi]))$$

we get

$$\bar{\sigma}([\xi]) = -[\xi] \text{ and } [\xi] \neq -[\xi] \text{ in } End_{\mathcal{M}_{rat}}(t_2(X))$$

because otherwise we would get $t_2(X) = 0$. Hence Theorem 4 does not hold true.

Following the example in Remark 3 we now consider the case of a complex K3 surface X with a non-symplectic group G acting trivially on the algebraic cycles. Any automorphism g of X preserves the 1-dimensional vector space $H^{2,0} = H^2(X, \Omega_X^2) \simeq \mathbf{C}\omega$. Hence g is non-symplectic iff there exists a complex number $\alpha(g) \neq 1$ such that $g^*(\omega) = \alpha(g)\omega$. Let $NS(X)$ and T_X be the lattices of algebraic and transcendental cycles on X . X is said to be unimodular if $\det T_X = \pm 1$. Let H_X be the finite cyclic group defined as the kernel of the map $Aut(X) \rightarrow O(NS(X))$, where $O(NS(X))$ denotes the group of isometries of $NS(X)$. Then there are only finitely many values for $m = |H_X|$. By [LSY Th. 5] one has the following result.

Theorem 5. *Let X be a complex K3 surface X with a non-symplectic group G acting trivially on the algebraic cycles. Let $m = |H_X| \neq 3$: then there exists a surjective morphism $F_n \rightarrow X$, where $F_n \subset \mathbf{P}^3$ is the Fermat surface, of degree $n \geq 4$*

$$F_n : X_0^n + X_1^n + X_2^n + X_3^n = 0$$

Here $n = m$ if X is unimodular and $n = 2m$, if X is not unimodular.

Corollary 2. *Let X be a complex K3 surface with a non-symplectic group G acting trivially on the algebraic cycles. Let $m = |H_X| \neq 3$. Then the motive of X is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties. K3 surfaces satisfying these conditions have $\rho(X) = 2, 4, 6, 10, 12, 16, 18, 20$.*

Proof. From Theorem 5 there is surjective morphism $F_n \rightarrow X$, with F_n a Fermat surface. By [SK] the motive $h(F_n)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties. By [Ki 6.6 and 6.8] if $f : Z \rightarrow X$ is a surjective morphism of smooth projective varieties, then $h(X)$ is a direct summand of $h(Z)$. Therefore $h(X)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties. The computation of the rank $\rho(X)$ appears in [LSY Th. 1 and Th. 2]. \square

4. EXAMPLES

In this section we describe some examples of K3 surfaces with a Nikulin involution i , such that $t_2(X) \simeq t_2(Y)$. Hence $h(X) \simeq h(Y)$. We will use the classification given by Van Geemen and Sarti in [VG-S] and by Garbagnati and Sarti in [G-S]. Their results are based on the following Theorem.

Theorem 6. ([VG-S 2.2]) *Let X be K3 surface with $\rho(X) = 9$ and a Nikulin involution i . Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ which we may assume to be ample. Let*

$$\Lambda_{2d} = \mathbf{Z}L \oplus E_8(-2)$$

Then, if $L^2 \equiv 2 \pmod{4}$, we have $\Lambda_{2d} = NS(X)$. If $L^2 \equiv 0 \pmod{4}$ we have either $NS(X) \simeq \Lambda_{2d}$ or $NS(X) \simeq \Lambda_{\overline{2d}}$. Here $\Lambda_{\overline{2d}}$ is the unique even lattice containing Λ_{2d} with $\Lambda_{\overline{2d}}/\Lambda_{2d} \simeq \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\Lambda_{\overline{2d}}$. For every $\Gamma = \Lambda_{2d}$, with $d > 0$ or $\Gamma = \Lambda_{\overline{2d}}$ with $d = 2m > 0$, there exists a K3 surface with a Nikulin involution i such that $NS(X) = \Gamma$ and $(H^2X, \mathbf{Z})^i{}^\perp \simeq E_8(-2)$.

Let's consider the following cases described in [VG-S]:

- (i) X is a double cover of \mathbf{P}^2 branched over a sextic curve and Y a double cover of a quadric cone in \mathbf{P}^3 ;
- (ii) X is a double cover of a quadric in \mathbf{P}^3 and Y is the double cover of \mathbf{P}^2 branched over a reducible sextic;
- (iii) the image of X under the map Φ_L is the intersection of 3 quadrics in \mathbf{P}^5 and Y is a quartic surface in \mathbf{P}^3 .

First we show that in the cases (i),(ii) and (iii) the map $f : X \rightarrow Y$ induces an isomorphism

$$t_2(X) \simeq t_2(Y)$$

Then, in Theorem 7 we prove that the same result holds if $g : X \rightarrow \mathbf{P}^1$ is a general elliptic fibration with a section and also Y is an elliptic fibration.

In the case (i) $NS(X) \simeq \mathbf{Z}L \oplus E_8(-2)$, with $L^2 = 2$ and $i^*L \simeq L$ (see [VG-S 3.2]). The map $\Phi_L : X \rightarrow \mathbf{P}^2$ is a double cover branched over a sextic curve C and $X/\langle i \rangle$ is a double cover of a quadric cone in \mathbf{P}^3 . Let σ denote the covering involution on X . Then $\sigma \neq i$. The quotient surface $Y = X/\langle \sigma \rangle$ is isomorphic to \mathbf{P}^2 . Let $j = \sigma \circ i = i \circ \sigma$ and let $G = \langle 1, \sigma, i, j \rangle \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. The quotient surfaces $\mathbf{P}^2 = X/\langle \sigma \rangle$ and $S = X/\langle j \rangle$ are both rational, because S is a Del Pezzo surface of degree 1 (see [VG-S 3.2]). The motives $h(\mathbf{P}^2)$ and $h(S)$ have no transcendental part. Therefore $t_2(X) \neq t_2(Y)$ and $t_2(X) \neq t_2(S)$, because $t_2(X) \neq 0$ for a K3 surface. From the identities in $End_{\mathcal{M}_{rat}}(t_2(X))$

$$[\xi] = 1/2([\xi] + \bar{\sigma}([\xi]) + 1/2([\xi] - \bar{\sigma}([\xi]))$$

$$[\xi] = 1/2([\xi] + \bar{j}([\xi]) + 1/2([\xi] - \bar{j}([\xi]))$$

we get, by Corollary 1 (iv), $[\xi] + \bar{\sigma}([\xi]) = [\xi] + \bar{j}([\xi]) = 0$ in $End_{\mathcal{M}_{rat}}(t_2(X))$. We have

$$\Psi(\Gamma_i) = \Psi_X((1 \times i)\Delta_X) = \Psi_X((1 \times \sigma \circ j)\Delta_X) = \Psi(\Gamma_{\sigma \circ j})$$

Therefore the class of $\bar{i}([\xi])$ in $End_{\mathcal{M}_{rat}}(t_2(X))$ equals $\bar{\sigma}([\xi]) \circ \bar{j}([\xi]) = (-[\xi]) \circ (-[\xi]) = ([\xi])^2 = [\xi]$ because $[\xi]$ is the identity of $End_{\mathcal{M}_{rat}}(t_2(X))$. Hence

$$\bar{i}([\xi]) - [\xi] = 0 \text{ in } End_{\mathcal{M}_{rat}}(t_2(X))$$

From Corollary 1 (ii) we get $\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$.

The proof for (ii) is similar to the previous one. In this case the lattice $\mathbf{Z}L \oplus E_8(-2)$ has index 2 in $NS(X)$ and we may assume that $NS(X)$ is generated by L , $E_8(-2)$ and $E_1 = (L + v)/2$, with $v \in E_8(-2)$, such that $v^2 = -4$. Then E_1 and E_2 , where $E_2 = (L - v)/2$, are the classes of 2 elliptic fibrations. The map

$$\Phi_L : X \rightarrow \mathbf{P}^3$$

is a 2:1 map to a quadric Q in \mathbf{P}^3 and it is ramified on a curve C of bidegree $(4, 4)$ ([VG-S 3.5]). The quadric Q is smooth, hence it is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. The covering involution $\sigma : X \rightarrow X$ of $X \rightarrow Q$ and the Nikulin involution i commute, the elliptic pencils E_1 and E_2 are permuted by i because $i^*L = L$ and $i^*v = -v$. i induces an involution i_Q on $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ which acts sending a point $\{(s, t), (u, v)\}$ to $\{(u, v), (s, t)\}$. The quotient $Q/\langle i_Q \rangle$ is isomorphic to \mathbf{P}^2 . Let $j = i \circ \sigma = \sigma \circ i$ in $Aut(X)$ and let $G = \{1, \sigma, i, j\} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. $S = X/\langle j \rangle$ is a Del Pezzo surface of degree 2, by [VG-S 3.5]. The

motives $h(Q)$ and $h(S)$ have no transcendental part, hence from the same argument as in (i), we get an isomorphism $t_2(X) \simeq t_2(Y)$.

We now consider the description given in [VG-S 3.7] of (iii). Let Y be the desingularization of the quotient surface $X / \langle i \rangle$. In this case there is a line bundle $M \in NS(Y)$ such that $\beta^*L \simeq f^*M$ and

$$H^0(X, L) \simeq f^*(H^0(Y, M)) \oplus f^*(H^0(Y, M - C))$$

where $\beta : \tilde{X} \rightarrow X$ is the blow-up at the 8 fixed points P_1, P_2, \dots, P_8 of i , $f : \tilde{X} \rightarrow Y$ and $C = (\sum_{1 \leq i \leq 8} C_i)/2 \in NS(Y)$, with C_i the rational curves on Y corresponding to the 8 singular points Q_1, \dots, Q_8 of $X / \langle i \rangle$. The above decomposition is the decomposition of $H^0(X, L)$ into the i^* eigenspaces. We have $L^2 = 8$, $M^2 = 4$, $h^0(M) = 4$, $h^0(M - C) = 2$ so that

$$\Phi_L : X \rightarrow \mathbf{P}^5 ; \Phi_M : Y \rightarrow \mathbf{P}^3 ; \Phi_{M-C} : Y \rightarrow \mathbf{P}^1$$

The image of X under Φ_L is the intersection of 3 quadrics in \mathbf{P}^5 and the involution i is induced by the involution

$$\mathbf{C}^6 : (x_0, x_1, x_2, x_3, y_0, y_1) \rightarrow (x_0, x_1, x_2, x_3, -y_0, -y_1)$$

The fixed points (P_1, P_2, \dots, P_8) lie in $X \cap \{y_0 = y_1 = 0\}$. The quadrics in the ideal of X are of the form

$$y_0^2 = Q_1(x), y_0 y_1 = Q_2(x), y_1^2 = Q_3(x)$$

where $x = (x_0, x_1, x_2, x_3)$. The line

$$l : x_0 = x_1 = x_2 = x_3 = 0$$

in \mathbf{P}^5 is fixed under i and $l \cap X = \emptyset$. The image of Y by Φ_M is the projection of X from the invariant line to the invariant \mathbf{P}^3 which is defined by $y_0 = y_1 = 0$. The image is the quartic surface in \mathbf{P}^3 defined by

$$Q_1(x)Q_3(x) - Q_2^2(x) = 0$$

which can be identified with Y .

We now use a result in [Vois 1.18] : if X is the K3 surface obtained as the intersection of 3 quadrics in \mathbf{P}^5 which are invariant under the involution

$$i : (x_0, x_1, x_2, x_3, y_0, y_1) \rightarrow (x_0, x_1, x_2, x_3, -y_0, -y_1)$$

then $i^*(\omega) = \omega$ for $\omega \in H^{2,0}(X)$ and i acts trivially on $A_0(X)$. Therefore, by Corollary 1 (iii) we get an isomorphism $\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$.

Next we consider the case of a K3 surface X which has an elliptic fibration $g : X \rightarrow \mathbf{P}^1$ with a global section $\sigma : \mathbf{P}^1 \rightarrow X$. The set of sections of g is the Mordell-Weil group MW_g with identity element σ .

MW_g is the subgroup of $\text{Aut } X$ consisting of all automorphism acting on a general fiber as translations and these translations preserve the holomorphic two form on X . Therefore, if there is an element τ of order 2 in MW_g then the translation by τ defines a Nikulin involution i on X .

Theorem 7. *Let X a general elliptic fibration $g : X \rightarrow \mathbf{P}^1$ with sections σ, τ as above. Let i be the corresponding Nikulin involution on X and let Y be the desingularization of $X / \langle i \rangle$. Then the map $f : X \rightarrow X / \langle i \rangle$ induces an isomorphism*

$$\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$$

Proof. In [VG-S 4.2] it is shown that for a general elliptic fibration $g : X \rightarrow \mathbf{P}^1$ there is an isomorphism $MW_g = \{\sigma, \tau\} \simeq \mathbf{Z}/2\mathbf{Z}$ where $\sigma : \mathbf{P}^1 \rightarrow X$. Hence the translation by τ defines a Nikulin involution i on X . The Weierstrass equation of X can be put in the form

$$X : y^2 = x(x^2 + a(t)x + b(t))$$

where the degree of $a(t)$ and $b(t)$ are 4 and 8 respectively. There are 8 singular fibers of type I_1 , which are rational curves with a node, corresponding to the zeroes $\{a_1, \dots, a_8\}$ of $a^2(t) - 4b(t)$ and 8 singular fibers of type I_2 , which are union of two \mathbf{P}^1 meeting in 2 points, corresponding to the zeroes $\{b_1, \dots, b_8\}$ of $b(t)$. The fixed points of the translation by τ are the 8 nodes in the I_1 -fibers. τ acts on the generic fiber E_t as the translation by a point P of order 2, i.e. $\tau(Q) = Q + P$ with $2P = 0$. The desingularization Y of the quotient surface $X / \langle \tau \rangle$ is an elliptic fibration with Weierstrass equation

$$Y : y^2 = x(x^2 - 2a(t)x + 9a(t)^2 - 4b(t)).$$

The generic fiber F_t of Y is the elliptic curve $E_t / \langle P \rangle$, where E_t is the generic fiber on X . Let

$$V = \bigcup_{t \in A} g^{-1}(t) = \bigcup E_t$$

where $A = \mathbf{P}^1 - \{a_1, \dots, a_8, b_1, \dots, b_8\}$. Then V is open in X and, for every point $x \in V$, the involution τ acts as translation by a point of order 2 on E_t , so that $2\tau(x) = 2x$. Therefore $2(1 \times \tau)(x, x) = (2x, 2x)$, for all $x \in V$ i.e. $(1 \times \tau)\Delta_V = \Delta_V$ with $\Delta_V = \Delta_X \cap (V \times X)$. We get $(1 \times \tau)\Delta_X - \Delta_X = 0$ on $V \times V$, hence $(1 \times \tau)\Delta_X - \Delta_X \in \mathcal{J}(X)$, with $\mathcal{J}(X) = \text{Ker } \Psi_X$ and $\Psi_X : A^2(X \times X) \rightarrow \text{End}_{\mathcal{M}_{rat}}(t_2(X))$. Therefore $\Psi_X(1 \times \tau)\Delta_X = \Psi_X(\Delta_X)$ and

$$\bar{\tau}([\xi]) = [\xi]$$

in $End_{\mathcal{M}_{rat}}(t_2(X))$, where ξ is the generic point of X . From Corollary 1 (ii) we get

$$\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$$

□

Remark 4. By [VG-S 4.1], if X is as in theorem 7, then the Neron-Severi group $NS(X)$ has rank $\rho(X) = 10$, and $\dim T_{X,\mathbf{Q}} = 12$ is even. In this case the isomorphism of Hodge structures $\phi_i : T_{X,\mathbf{Q}} \simeq T_{Y,\mathbf{Q}}$, induced by the involution i , is an isometry. On the contrary, in the cases described in (i),(ii), (iii), where $\rho(X) = 9$, ϕ_i is not an isometry. This follows from [VG-S 2,5] because $\dim T_{X,\mathbf{Q}}$ is odd.

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